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A New Upper Bound for the Isoperimetric Number of deBruijn Networks

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Abstract—Delorme and Tillich found an upper bound and a lower bound for the isoperimetric number $i^n(d)$ of deBruijn Networks over the alphabet $\{0, 1, \dots, d-1\}$ using eigenvalue techniques (see [1]). We improve their upper bound for $i^n(d)$ and give constructions for the sets of vertices of the deBruijn Network, which lead to our bound.

Keywords—Isoperimetric number, deBruijn Network.

1. INTRODUCTION

The **d-ary deBruijn Network $\{0, 1, \dots, d-1\}^n_{dB}$** is an undirected graph without loops and without multiple edges. The set of vertices V^n consists of all n-tuples over the alphabet $\{0, 1, \dots, d-1\}$. Let $u, v \in V^n$ with $u \neq v$. Assume that $v = [v_0, \dots, v_{n-1}]$.

The vertices u and v are connected by an edge, if

$$u \in \{[b, v_0, \dots, v_{n-2}], [v_1, \dots, v_{n-1}, b] \mid b \in \{0, 1, \dots, d-1\}\}.$$

We define the **isoperimetric number $i^n(d)$ of the deBruijn Network $\{0, 1, \dots, d-1\}^n_{dB}$** by

$$i^n(d) := \min_{\substack{S \subset V^n \\ 1 \leq |S| \leq |V^n|/2}} \frac{|N(S)|}{|S|},$$

where $N(S)$ is the set of vertices of $V^n \setminus S$ which are connected to some vertex in S.

2. A CONSTRUCTION OF SETS WITH SMALL NEIGHBOURHOODS

It is convenient to compress the elements in V^n in the following way: a substring with a constant symbol is represented by the symbol and by the length of this substring. For instance,

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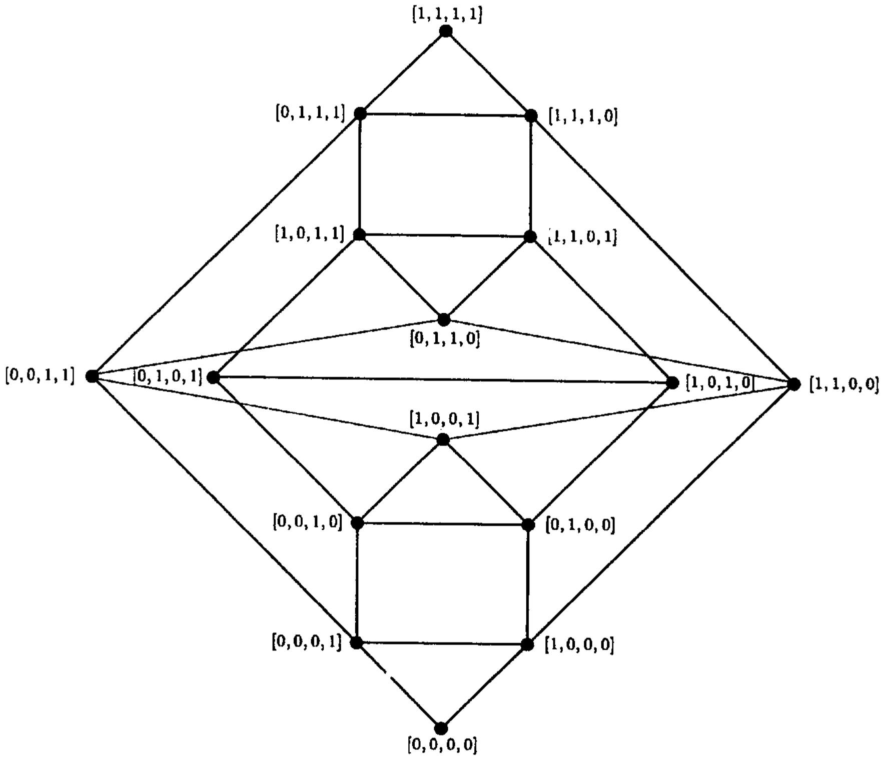

Figure 1. The $\{0,1\}^4_{dB}$.

$[0,0,\dots,0,0] \in V^n$ becomes $[0^n]$, and $[1,0,\dots,0,1] \in V^n$ becomes $[1,0^{n-2},1]$. By "$*$", we denote any element of $\{0,1,\dots,d-1\}$, and by "$+$", we denote any element of $\{1,\dots,d-1\}$. For instance if $d=3$ and $n=3$, then $[+,0,*] = \{[1,0,0],[1,0,1],[1,0,2],[2,0,0],[2,0,1],[2,0,2]\}$. With the help of this notation we formulate the following two theorems.

THEOREM 2.1. *Consider the deBruijn Network $\{0,\dots,d-1\}^n_{dB}$ for $n = 4m+1$ and $m \geq 2$. Let S be a subset of V^n with*

$$S = \bigcup_{k=0}^{m-1} \left[*^{m+k}, +, 0^{2m}, *^{m-k}\right] \cup \left[*^m, 0^{2m+1}, *^m\right] \cup \bigcup_{k=0}^{m-1} \left[*^{m-k}, 0^{2m}, +, *^{m+k}\right].$$

Then $|S| = d^{2m} + 2m(d-1)\,d^{2m}$ and $|N(S)| = 2(d-1)\,d^{2m} - (d-1)$.

PROOF. Without loss of generality, it suffices to check only three cases.

CASE 1.

$$N\left(\left[*^m, 0^{2m+1}, *^m\right]\right) \subset \underbrace{\left[*^{m-1}, 0^{2m+1}, *^{m+1}\right]}_{\subset S} \cup \underbrace{\left[*^{m+1}, 0^{2m+1}, *^{m-1}\right]}_{\subset S}.$$

CASE 2. Let $0 \leq k \leq m-2$. Then

$$N\left(\left[*^{m+k}, +, 0^{2m}, *^{m-k}\right]\right) \subset \underbrace{\left[*^{m+k-1}, +, 0^{2m}, *^{m-k+1}\right]}_{\subset S} \cup \underbrace{\left[*^{m+k+1}, +, 0^{2m}, *^{m-k-1}\right]}_{\subset S}.$$

CASE 3.

$$N\left(\left[*^{2m-1}, +, 0^{2m}, *\right]\right) \subset \underbrace{\left[*^{2m-2}, +, 0^{2m}, *^2\right]}_{\subset S} \cup \left[*^{2m}, +, 0^{2m}\right].$$

According to the symmetry of S, we have $N(S) = [*^{2m}, +, 0^{2m}] \cup [0^{2m}, +, *^{2m}]$. Since $[0^{2m}, +, *^{2m}] \cap [*^{2m}, +, 0^{2m}] = [0^{2m}, +, 0^{2m}]$, we obtain $|N(S)| = (d-1)\,d^{2m} + (d-1)\,d^{2m} - (d-1) = 2(d-1)\,d^{2m} - (d-1)$. ∎

THEOREM 2.2. *Consider the deBruijn Network* $\{0,\dots,d-1\}^n_{dB}$ *for* $n = 4m+1+f$, *for* $f \in \{1,2,3\}$ *and* $m \geq 2$. *Let* S *be a subset of* V^n *with*

$$S = \bigcup_{k=0}^{m-1} \left[*^{m+k}, +, 0^{2m+f}, *^{m-k}\right] \cup \left[*^{m}, 0^{2m+f+1}, *^{m}\right] \cup \bigcup_{k=0}^{m-1} \left[*^{m-k}, 0^{2m+f}, +, *^{m+k}\right].$$

Then $|S| = d^{2m} + 2m(d-1)\,d^{2m}$ *and* $|N(S)| = 2(d-1)\,d^{2m}$.

PROOF. With the same calculation as in the proof of Theorem 2.1, we obtain $N(S) = [*^{2m}, +, 0^{2m+f}] \cup [0^{2m+f}, +, *^{2m}]$. Since $[0^{2m+f}, +, *^{2m}] \cap [*^{2m}, +, 0^{2m+f}] = \emptyset$, we get $|N(S)| = (d-1)\,d^{2m} + (d-1)\,d^{2m} = 2(d-1)\,d^{2m}$. ∎

3. A NEW UPPER BOUND FOR THE ISOPERIMETRIC NUMBER OF DEBRUIJN NETWORKS

Delorme and Tillich used the spectrum of deBruijn Networks to calculate upper and lower bounds for the isoperimetric number[1] (see [1]). They have shown that the following inequalities hold:

$$\frac{1}{n} \leq i^n(d) \leq \frac{2\sqrt{d}\pi}{(n+1)\sqrt{1-2d\pi^2/(n+1)^2}}.$$

Their proof is a proof of existence, but not constructive. With the constructions of Section 2, we are able to improve their upper bound.

We have $|S| = d^{2m}+2m(d-1)\,d^{2m} \leq d^n/2$. If $n = 4m+1$, we get $|N(S)| = 2(d-1)\,d^{2m}-(d-1)$. Using this equality we get the following quotient:

$$\begin{aligned}\frac{|N(S)|}{|S|} &= \frac{2(d-1)\,d^{2m}-(d-1)}{d^{2m}+2m(d-1)\,d^{2m}} \\ &\leq \frac{2(d-1)\,d^{2m}}{d^{2m}+2m(d-1)\,d^{2m}} \\ &= \frac{2(d-1)}{1+2m(d-1)} \\ &\leq \frac{2(d-1)}{2m(d-1)} \\ &= \frac{2}{2m} \\ &= \frac{1}{m}.\end{aligned}$$

Substitution of m by $(n-1)/4$ yields $|N(S)|/|S| \leq 1/((n-1)/4) \leq 4/(n-1)$. If $n = 4m+1+f$, we get $|N(S)| = 2(d-1)\,d^{2m}$. We have

$$\begin{aligned}\frac{|N(S)|}{|S|} &= \frac{2(d-1)\,d^{2m}}{d^{2m}+2m(d-1)\,d^{2m}} \\ &= \frac{2(d-1)}{1+2m(d-1)} \\ &\leq \frac{2(d-1)}{2m(d-1)} \\ &= \frac{2}{2m} \\ &= \frac{1}{m}.\end{aligned}$$

Substitution of m by $(n-1-f)/4$ yields $|N(S)|/|S| \leq 1/((n-1-f)/4) \leq 4/(n-4)$. Summarizing the calculations above yields our main result.

[1]They call the isoperimetric number magnifying coefficient.

THEOREM 3.1. *Consider the deBruijn Network* $\{0,1,\dots,d-1\}$ *for* $n \geq 9$. *Then*

$$i^n(d) \leq \frac{4}{n-4}.$$

In contrast to Delorme and Tillich, we have explicitly constructed the sets which lead to our upper bound.

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